DSS Receiver Thermal Noise Model Comparisons

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DSN receiver thermal noise is a component of the Doppler jitter noise measured during DSN system testing. This thermal noise has been modelled theoretically through an evolution of approximations over a period of 20 years. The model is required to predict the expected test result. This article compares the two latest and most accurate models in order to determine if changes in the test algorithm are required for desired accuracy. Also introduced is a new and highly simplified model that exhibits differentials of the same order as the more complicated algorithms. The study concluded that the three models are indistinguishable within the nominal operating range of the receivers.

I. Introduction

DSN receiver thermal (phase) noise is one contribution to overall system noise, as measured during DSN system testing. The test is mechanized to extract the noise as "Doppler jitter," sampled to include the majority of the frequency spread, and reduced finally by the Allen variance technique to reject long-term drift effects.

The receiver thermal contribution is predominant at lower signal levels, but tends to be "swamped out" at strong signal levels by VCO noise, quantization noise, and similar sources.

However, for "pass-fail" criteria, a reasonable theoretical model of expected thermal noise vs. signal level is required across the entire range. Variations of this model have developed historically over more than 20 years. In order to consider possible present "update" of the existing test model, this study was conducted. It compares the two most "recent" models, and introduces a simplified exponential model with comparable accuracy. These thermal models are, of course, variance-added to models of other contributions to obtain final estimates for total system Doppler jitter evaluation.

II. Receiver Thermal Noise Major Parameters

The prime parameter in DSN receiver design is a quantity labelled $2\,BL_o$. This number, displayed on equipment pushbuttons for selection, represents the square-and-double-sided phase noise bandwidth if the loop operated entirely linearly, without detector signal-to-noise ratio (SNR) gain, and if the output SNR (design point) variance were unity. $2\,BL_o$ is, therefore, entirely hypothetical, for these conditions are inexact. However, loop filter time constants and gain calculations are determined for specified values of $2\,BL_o$, and all equipment is structured by this design parameter, with conventional linear optimization.

Now, given $2\,BL_o$ and a system operating temperature, an exact (minimum) reference *signal level* can be expressed. It is the level when signal and noise power are equal and if the noise bandwidth *is* exactly $2\,BL_o$, as referred to the receiver input. Said another way, it is the signal that would exhibit an (undetected) phase variance of unity if passed through a filter (double-sided) with a width of $2\,BL_o$ Hz (open loop). This is the reference level used in DSN testing. It has the value:

$$W_o = KT(2\,BL_o)$$

$$W_o \text{ (in dBm)} = 10 \log (T) + 10 \log (2\,BL_o) - 198.6$$

W_o = reference signal level, W or dbm

T = system temperature, K

K = Blotzman's constant, W/K/Hz

$2\,BL_o$ = reference (loop) noise bandwidth

In some literature, W_o is 3 dB less, $W_o/2$, at a point where the hypothetical "linear-loop" variance is unity. This discussion avoids that definition for conformance with the DSN testing approach, and as a common base in model comparison.

Using the above, W_o in this discussion takes the role of "minimum signal level" to be investigated. This is reinforced by the fact that *actual* receiver thermal noise, all factors included, results in a variance very close to unity at W_o. This is, therefore, in the vicinity of the original definition of threshold, where the receiver is no longer reliably phase-stable; a true minimum.

Results of this study are presented as jitter (standard deviation, deg, the square-root of variance) vs. margin, dB for various models and $2\,BL_o$. Given W_o, margin (M) is easily defined by the ratio of actual input signal level to W_o:

$$M = \frac{W}{W_o}$$

$$M \text{ (dB)} = 10 \log \frac{W}{W_o}$$

$o \leqslant M \text{ (dB)} \leqslant 50$ (model spread)

W = actual signal level, W or dBm

Also:

$$M \text{ (dB)} = W\text{(dBm)} - W_o \text{ (dBm) (for test settings).}$$

III. Predetection Signal-to-Noise Ratio

The phase-lock-loop predetection limiter effects are functions of the predetection signal-to-noise ratio, or PH in most literature. These effects are normally isolated as two-fold: the signal suppression factor, α (signal gain), and the limiter performance factor, Γ (SNR transfer). Both effects are strictly functions of PH; the Γ factor (EXACT) model is quite complicated. Model expressions for these effects are given in the Appendix. However, these expressions are invalid without a sound definition of PH itself.

Since PH occurs at a predetection point, and can be considered as due entirely to input levels, it is easily defined from the predetection filter noise bandwidth, and previous notations:

$$PH = \frac{W}{KT(BC)} = \frac{W_o}{KT(BC)} \times \frac{W}{W_o}$$

$$= \frac{KT(2\,BL_o)}{KT(BC)} \times M = M \times \frac{2\,BL_o}{BC}$$

BC = Predetection (two-sided) noise bandwidth, Hz.

IV. The System Signal-to-Noise Ratio Conversion

This concept, introduced by Yuen, describes a very clear-cut model approach. It states that expected output thermal variance (as on Doppler measures), can be modelled by two discrete and simple steps, independent of the quantity of nonlinear (sinusoidal) detectors in the signal path:

(1) Determine the total linear theory variance to output; just consider all detectors linear. Double SNR (divide variance by two) to account for detection enhancement. The figure obtained is:

$$\sigma_\ell^2 = \sigma_\ell^2 (M) = 1/p$$

p = system SNR ratio

σ_ℓ^2 = system phase variance (linear computation)

(2) Convert the system linear variance from the Gaussian to nonlinear modulo 2-Π (sinusoidal detector) value in a sinusoidal conversion spectral density expression. Yuen proved this valid.

These steps are used for the organization of both the EXACT and approximate models.

V. Expected Doppler Receiver Thermal Noise, *E* (σ)

The general algorithm for thermal noise, using conventional loop design by Jaffee/Rechtin can be organized as follows:

Inputs:

$2\,BL_o$ = Design point loop noise bandwidth, Hz

BC = Predetection noise bandwidth, Hz

T = System temperature, K

W(dBm) = Input signal level, dBm.

Prime quantities (all models):

$$W_o(\text{dBm}) = 10\log(T) + 10\log(2\,BL_o) - 198.6$$

$$M(\text{dB}) = W(\text{dBm}) - W_o\,(\text{dBm})$$

$$M = 10^{\frac{M(\text{dB})}{10}} \qquad 1 < M < 10^5$$

$$PH = \frac{BC}{2BL_o} \times M.$$

The algorithm proper follows two major steps:

STEP NO. 1:

$$\sigma_\ell^{\,2} = \frac{1}{p} = \frac{1}{2M} \times \left[\frac{1}{3} + \frac{2}{3}\frac{\alpha}{\alpha_o}\right] \times \Gamma = \text{system variance (linear computation)}$$

$$\left.\begin{aligned} \alpha &= F(PH)\,|_M \\ \alpha_o &= F(PH @ M = 1) \\ \Gamma &= G(PH)\,|_M \end{aligned}\right\}\ \text{Expressions vary among models. See the Appendix.}$$

STEP NO. 2:

$$E(\sigma) = H_1\,(\sigma_\ell^{\,2})\ \text{(Simon/Lindsey/Yuen) (EXACT)}$$

$$H_2\,(\sigma_\ell^{\,2}, \alpha/\alpha_o)\,|_M\ \text{(Tausworthe) (approximate)}$$

The algorithm, thus, involves the calculation of W_o, M, PH, α, Γ, $\sigma_\ell^{\,2}$, and $E(\sigma)$. The first three are elementary, requiring compatible interpretation only. The last three are model-peculiar and somewhat complicated. The factor Γ is not presently available expressed in exact form.

Three models of the above are given in detail in the Appendix:

(1) The original Jaffee-Rechtin linear model

(2) The Tausworthe linear spectral analysis model (in use for DSN testing)

(3) The Simon/Lindsey/Yuen EXACT Model

The first model, linear, is plotted throughout the literature and this plot will not be repeated. The second model, on a straight plot, was indistinguishable from the third, or EXACT model. This latter model is plotted, in the range of interest, in Fig. 1.

Using this EXACT model as a standard, the differential between it and the second, or Tausworthe (presently in use within the testing algorithm), is shown in Fig 2, as a function of margin. Except for a 2 degrees difference at low margin (M = 1), the two models agree within 0.5 deg, with a small bias. The RMS of the differential is 0.4 deg, two bandwidth ratios, inclusive. If the 2 deg differential near M = 1 is disregarded, the overall RMS differential drops to 0.18 deg.

There appears to be little point to changing the DSN algorithm to the EXACT model, for differentials on the order of those above are essentially "masked" by other noise contributions in the system.

However, there is an elementary approximation to the EXACT model, developed by the author; the description follows. It might be worthwhile to use this in testing for algorithm simplicity.

VI. The Exponential Approximation Model

A quick glance at Fig. 1 suggests that the full model, with decibel abscissa and standard deviation ordinate, appears to approach the form of a simple exponential decay.

When this form was tried, with coefficients to fit the end points (margin 0 dB and 50 dB), the resulting exponential fit was within 3 deg throughout. The error plot had the form of the Gaussian derivative. This led to the following two-term exponential model:

$$E(\sigma) = \underbrace{55.12\, e^{-a_0 \times M}}_{\text{Main Approximation}} + \underbrace{M \times a_1 e^{-a_2 \left(\frac{M}{10}\right)^2}}_{\text{Trim Term}}$$

M = Margin in dB

A variation-of-parameters closure program was applied to (1), vs. the EXACT model, and minimum RMS differential for the receiver bandwidth ratios was obtained (see Table 1). The deviations of these approximations from the EXACT model is shown on Fig. 3. The overall RMS of the deviations is 0.22 deg. This level, as with the Tausworthe model, would normally be masked by other contributions.

V. Conclusions

Three nonlinear models for receiver thermal noise were investigated. Within the range of interest (0 dB to 50 dB margin), differences between the models were indistinguishable.

One of the models, the Tausworthe or linear spectral density approximation, is presently in use for DSN system testing. There appears to be little practical advantage to altering the algorithm at this time.

Another model, the Simon/Lindsey, or EXACT algorithm, is complicated to program, but generally considered the most accurate noise prediction available. It might be reasonable to incorporate this model at some future date during general software update.

Finally, the third model, or Exponential Approximation, by the author, is by far the simplest expression for thermal noise that is presently on hand. It is seen as useful in abbreviated programming and, possibly, as a future general-purpose design and test-predictive expression, replacing even the EXACT model when simplicity is desired.

Table 1. a_0, a_1, a_2, coefficients

Block	$BC/2\,BL_o$	(Typ)	a_o	a_1	a_2
III	166.7	2000/12	0.11288	0.58027	0.21658
IV	66.67	200/3	0.11198	0.55579	0.26151
IV	200	2000/10	0.11364	0.59611	0.21125

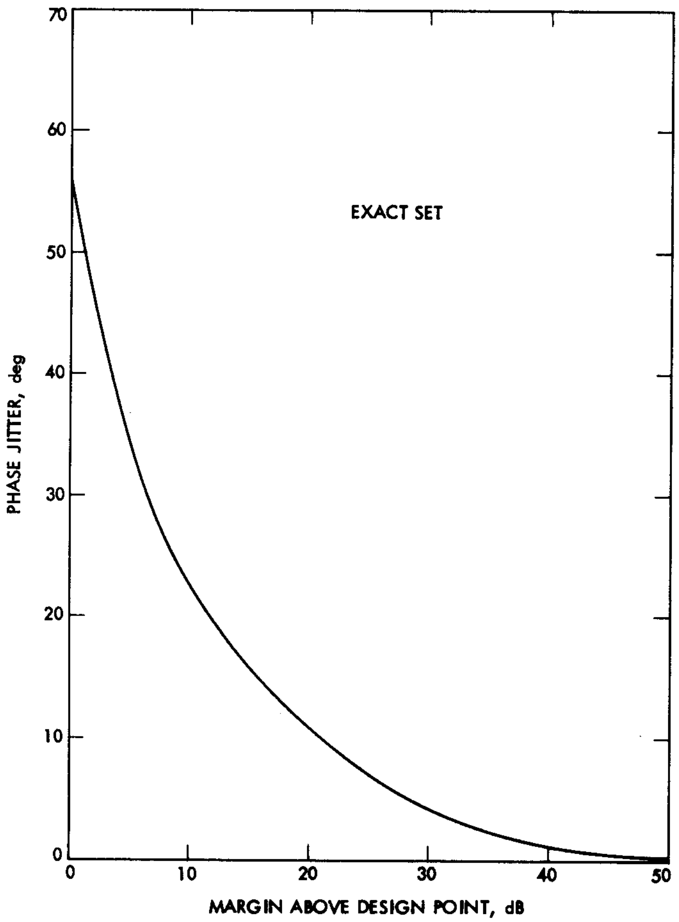

Fig. 1. Receiver thermal noise exact model

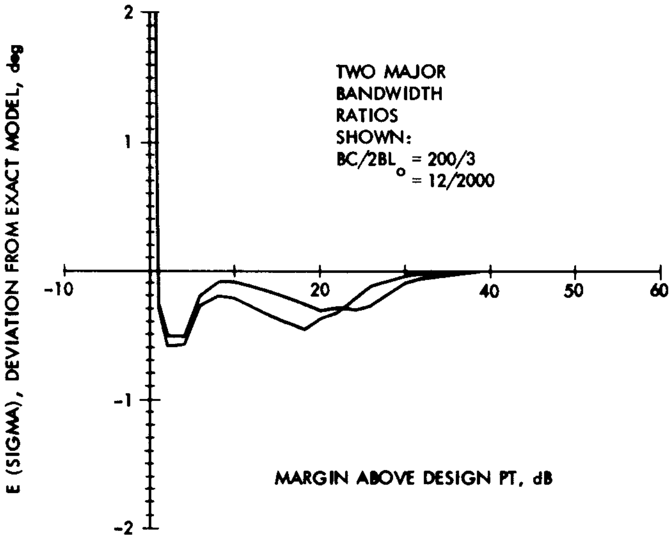

Fig. 2. Deviation between exact and linear-spectral-density approximation models

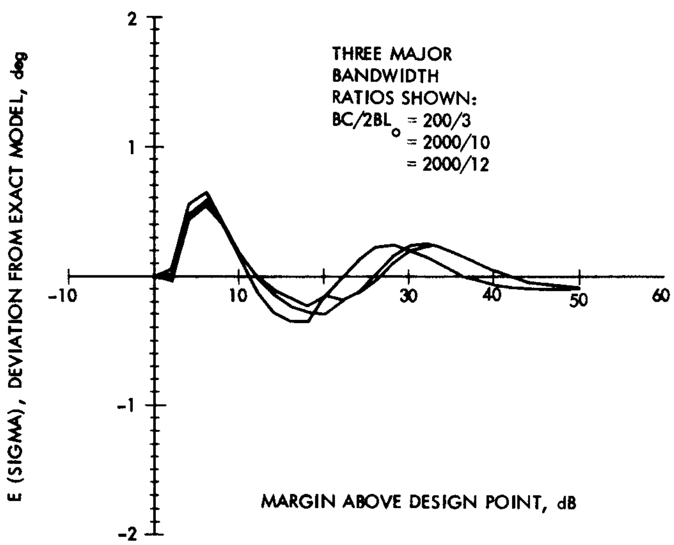

Fig. 3. Deviation between exact and exponential approximation models

Appendix
Receiver Thermal Noise Models

(A) Original Linear Model

Jaffee/Rechtin Linear Loop

(1) $\sigma_\ell^2 = \frac{1}{2M}\left[\frac{1}{3} + \frac{2}{3}\frac{\alpha}{\alpha_o}\right] \times \Gamma$

(2) $\frac{\alpha}{\alpha_o} = \left[\frac{1 + \frac{4}{\Pi} \times \frac{1}{PH}}{1 + \frac{4}{\Pi} \times \frac{1}{PH} \times \frac{1}{M}}\right]^{1/2}$

$PH = PH(1)$

(3) $\Gamma = 1$

(4) $E(\sigma) = \sigma_\ell$ (no conversion)

$\sigma^o = \frac{180}{\pi}\sigma_\ell$

This model applies in the strong-signal ($M > 20$ dB) range only. It considers the detector completely linear, and the limiter as a device that simply holds its power output constant. The model is not considered in the comparison. $E(\sigma)$ is in error by nearly 50 percent at unity margin. It does, however, provide criteria for DSN receiver design, which normally operates near strong-signal levels.

(B) Study Model No. 1

Tausworthe Nonlinear Approximations (used for DSN testing)

(1) $\sigma_\ell^2 = \frac{1}{2M} \times \left[\frac{1}{3} + \frac{2}{3}\frac{\alpha}{\alpha_o}\right] \times \Gamma$

(2) $\alpha = \left[\frac{0.7854\, PH + 0.4768\, PH^2}{1.0 + 1.024\, PH + 0.4768\, PH^2}\right]^{1/2}$

$\alpha_o = \alpha\,[PH(M = 1)]$

(3) $\Gamma = \frac{1 + PH}{0.862 + PH}$

(4) $\sigma_\ell^2 = \frac{\left(1 - e^{-a^2}\right)^2}{a^2}$

(5) $$(1 + r)\frac{\left[1 + \frac{2a^2\left(1 - e^{-a^2} - a^2 e^{-a^2/2}\right)}{r\left(1 - e^{-a^2}\right)^2}\right]^{1/2}}{\frac{1 + r\left(1 - e^{-a^2}\right)}{a^2}}$$

$r = 2\frac{\alpha}{\alpha_o} = r(M)\ \sigma_\ell^2 = \sigma_\ell^2\,(M)$

$\sigma^2 = \frac{\pi^2}{3}\left[1 - e^{-3a^2/\pi^2}\,(1 + 0.13a^2)\right]$

$\sigma^o\,(M) = \frac{180}{\pi}\,\sigma^2\,(M)$

NOTE: Parametric equation in α^2 until agreement to 10^{-4} in σ^2. Place σ_ℓ^2 in σ^2 for modulo 2π.

a^2 is the variance of a phase process and is iterated before being converted to modulo 2π. The algorithm is called linear spectral analysis approximation, and also uses algebraic approximations to EXACT expressions for α and Γ.

(C) Study Model No. 2

Simon/Lindsey and Yuen EXACT Conversion (near approximation)

(1) $\sigma_\ell^2 = \frac{1}{2M} \times \left[\frac{1}{3} + \frac{2}{3}\frac{\alpha}{\alpha_o}\right] \times \Gamma$

(2) $\alpha = \frac{\pi}{4}\,PH\,e^{-PH/2}\left[I_o\left(\frac{PH}{2}\right) + I_1\left(\frac{PH}{2}\right)\right]$

$\alpha_o = \alpha$ @ M = 1

$$(3)\ \Gamma = \frac{1 - e^{-PH}}{\frac{\pi}{4} PH \times e^{-PH} \times \left[I_o\left(\frac{PH}{2}\right) + I_1\left(\frac{PH}{2}\right)\right]\left[1 + \left(\frac{3.448}{\pi} - 1\right) e^{-PH(1 - \pi/4)}\right]}$$

(INEXACT)

$$(4) \qquad \rho = \frac{1}{\sigma_\ell{}^2}$$

$$E(\sigma^2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\phi^2 e^{\rho \cos(\phi)} d\phi}{I_o(\rho)}$$

$$= \frac{\pi^2}{3} + 4 \sum_{N=1}^{\infty} \left[(-1)^N \frac{I_n(\rho)}{I_o(\rho)} \cdot \frac{1}{N^2}\right]$$

$$\sigma^o = \frac{180}{\pi} \cdot \sigma$$

$$I_n(x) = \left(\frac{X}{2}\right)^2 \sum_{M=0}^{\infty} \frac{\left(\frac{X}{2}\right)^{2M}}{M!\ (M + N)!}$$

This is the most accurate model presently available. It was used as the standard for comparison.